JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 94, 525–535 (1983)

On Oscillation of Solutions of Higher Order Forced Functional Differential Inequalities

S. R. GRACE AND B. S. LALLI*

Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 0W0, Canada

Submitted by W. Welsh

Oscillation criteria for the class of forced functional differential inequalities $x(t)\{L_n x(t) + f(t, x(t), x[g_1(t)],..., x[g_m(t)]) - h(t)\} \leqslant 0$, for n even, and $x(t)\{L_n x(t) - f(t, x(t), x[g_1(t)],..., x[g_m(t)]) - h(t)\} \geqslant 0$, for n odd, are established.

1. INTRODUCTION

A great deal of literature exists on the oscillation and nonoscillation of the homogeneous equation

$$x^{(n)}(t) + H(t, x[g(t)]) = 0$$

and the nonhomogeneous equation

$$x^{(n)} + H(t, x[g(t)]) = h(t),$$

both for n even and n odd. For this see [2–8] and the references cited in them. However, not much is known about inequalities of the type

$$x(t)\{L_n x(t) + f(t, x(t), x[g_1(t)],..., x[g_m(t)]) - h(t)\} \leqslant 0 \qquad \text{for } n \text{ even} \qquad (1)$$

and

$$x(t)\{L_n x(t) - f(t, x(t), x[g_1(t)],..., x[g_m(t)]) - h(t)\} \geqslant 0 \qquad \text{for } n \text{ odd}, \qquad (2)$$

where $L_0 x(t) = x(t)$, $L_k x(t) = a_k(t)(L_{k-1} x(t))^{\cdot}$, $a_n(t) = 1$, $k = 1, 2,..., n$, $(^{\cdot} = d/dt)$.

Our main aim in this paper is to discuss the oscillatory behavior of the solutions of (1) (or (2)). We impose conditions on a_i, $i = 1, 2,..., n-1$ so

* The research of this author was supported partially by National Science Education and Research Council Grant A5293.

0022-247X/83 $3.00

that we can replace the often used statement "the bounded solution $x(t)$ of (1) (or (2))" by "the solution $x(t)$ of (1) (or (2)) such that

$$\lim_{t\to\infty} \frac{x(t)}{\alpha_2(t)} = 0, \qquad \alpha_2(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \frac{1}{a_2(s_2)}\, ds_2\, ds_1,"$$

and obtain new criteria for oscillation of solutions of (1) (or (2)).

Our results in Section 2 generalize and improve some of the results of the present authors [1], Kim [8], Lovelady [9], and Kartsatos and Manougian [7].

2. Main Results

In the sequel it is assumed that

$$a_k, g_i \in C[R_+ = [0, \infty),\ R_+\backslash\{0\}],\quad g_i(t) \to \infty \quad \text{as} \quad t \to \infty, \tag{3}$$

for $k = 1, 2,..., n-1$ and $i = 1, 2,..., m$;

$$\int^{\infty} \frac{1}{a_k(s)}\, ds = \infty, \qquad k = 1, 2,..., n-1; \tag{4}$$

$$\lim_{t\to\infty} \frac{1}{\alpha_2(t)} \sum_{j=1}^{k} c_j \alpha_j(t) > 0, \qquad \alpha_0(t) = 1, \tag{5}$$

for every choice of constants c_j with $c_k > 0$, $k = 2, 3,..., n-1$, where

$$\alpha_1(t) = \int_c^t \frac{1}{a_1(s)}\, ds,\ \alpha_k(t)$$
$$= \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \frac{1}{a_2(s_2)} \cdots \int_c^{s_{k-1}} \frac{1}{a_k(s_k)}\, ds_k\, ds_{k-1} \cdots ds_1,$$

$k = 1, 2,..., n-1$ and $c \geqslant 0$;

$$f \in C[R_+ \times R^{m+1}, R] \qquad \text{for} \quad x, x_i > 0, \quad i = 1, 2,... m \tag{6}$$

implies $f(t, x, x_1,..., x_m)$ is positive and nondecreasing with respect to x, x_i, $i = 1, 2,..., m$ for all $t \geqslant 0$;

$$f(t, x, x_1,..., x_m) \leqslant -f(t, -x, -x_1,..., -x_m), \tag{7}$$

for $x, x_i > 0$, $i = 1, 2, \ldots, m$ and all $t \geqslant 0$;

$$h \in C[R_+, R] \tag{8}$$

and there exists an oscillatory function $\eta(t)$ such that $L_n \eta(t) = h(t)$, $L_k \eta(t)$ is oscillatory and $L_k \eta(t) \to 0$ as $t \to \infty$ for $k = 0, 1, \ldots, n-1$.

The following Lemma is analogous to [1, Theorem 1]. We shall sketch the proof.

LEMMA. *Let conditions* (3)–(8) *hold. If x is a nontrivial solution of* (1) *(or* (2)*) such that*

$$x(t) > 0, \qquad x(t)/\alpha_2(t) \to 0 \quad as \quad t \to \infty, \qquad t \geqslant t_0 \geqslant 0,$$

then there is a $t^* \geqslant t_0$ such that $y(t) = x(t) - \eta(t)$ is a solution of (1) *(or* (2)*) for $t \geqslant t^*$ with the following properties*:

(i) *we have $y(t) > 0, y(t)/\alpha_2(t) \to 0$ as $t \to \infty$;*

(ii) *we have $(-)^{k-1} L_k y(t) > 0$, $L_k x(t) \cdot L_k y(t) > 0$ for $t \geqslant t^*$, $k = 1, 2, \ldots, n-1$;*

(iii) *we have $L_k x(t) \to 0$ monotonically as $t \to \infty$, $k = 2, 3, \ldots, n-1$.*

Proof. We only consider (1). Let $x(t)$ be a nontrivial solution of (1) such that $x(t) > 0$ for $t \geqslant t_0 \geqslant 0$ and $x(t)/\alpha_2(t) \to 0$ as $t \to \infty$. Choose $t_1 \geqslant t_0$ so that $g_i(t) \geqslant t_0$, $t \geqslant t_1$, $i = 1, 2, \ldots, m$. Thus $x[g_i(t)] > 0$ for $t \geqslant t_1$ and $i = 1, 2, \ldots, m$. Inequality (1) becomes

$$L_n x(t) + f(t, x(t), x[g_1(t)], \ldots, x[g_m(t)]) - h(t) \leqslant 0. \tag{9}$$

Let $y(t) = x(t) - \eta(t)$, then (9) becomes

$$\begin{aligned} L_n y(t) + f(, y(t) + \eta(t), y[g_1(t)] \\ + \eta[g_1(t)], \ldots, y[g_m(t)] + \eta[g_m(t)]) \leqslant 0. \end{aligned} \tag{10}$$

It is easy to check, using Kiguradze's lemma, that $L_k y(t)$ is of fixed sign for $t \geqslant t_1$ and $k = 0, 1, \ldots, n$. Now if $y(t) < 0$ for $t \geqslant t_2 \geqslant t_1$, then $y(t) + \eta(t) > 0$ implies $\eta(t) > -y(t) > 0$, a contradiction to the oscillatory character of $\eta(t)$. Hence $y(t) > 0$ for $t \geqslant t_2$. We write (10) as the following system $y = y_1$, $y_1' = y_2/a_1, \ldots, y_{n-1}' = y_n/a_{n-1}$ and $y_n'(t) \leqslant -f(t, y(t) + \eta(t), \ldots, y[g_m(t)] + \eta[g_m(t)])$. As in the proof of [1, Theorem 1], one can easily see that $y_n(t) > 0$ for $t \geqslant t_3 \geqslant t_0$. If $L_{n-1} x(t) < 0$ for $t \geqslant t_3$, then $L_{n-1} y(t) + L_{n-1} \eta(t) < 0$ implies $-L_{n-1} \eta(t) > L_{n-1} y(t) > 0$, a contradiction to the oscillatory character of $L_{n-1} \eta(t)$. It is now easy to conclude that $y_n(t) \to 0$ as

$t \to \infty$ for $n > 2$. If this were not the case, there would exist a constant $C > 0$ such that

$$y_n(t) > C \quad \text{for} \quad t \geqslant t_4, \qquad \text{for some} \quad t_4 \geqslant t_0.$$

However this implies that

$$\begin{aligned}
y(t) = y_1(t) > y_1(t_4) &+ y_2(t_4) \int_{t_4}^{t} \frac{1}{a_1(s)}\, ds + \cdots \\
&+ y_{n-1}(t_4) \int_{t_4}^{t} \frac{1}{a_1(s_1)} \int_{t_4}^{s_1} \cdots \\
&\times \int_{t_4}^{s_{n-3}} \frac{1}{a_{n-2}(s_{n-2})}\, ds_{n-2} \cdots ds_1 \\
&+ C \int_{t_4}^{t} \frac{1}{a_1(s_1)} \int_{t_4}^{s_1} \cdots \\
&\times \int_{t_4}^{s_{n-2}} \frac{1}{a_{n-1}(s_{n-1})}\, ds_{n-1} \cdots ds_1 \\
&= \sum_{i=0}^{n-2} y_{i+1}(t_4)\, \alpha_i(t) + C\alpha_{n-1}(t), \qquad \alpha_0(t) = 1.
\end{aligned}$$

Dividing the above inequality by $\alpha_2(t)$, taking the limit as $t \to \infty$ and using condition (5), we get a contradiction to the fact that $y_1(t)/\alpha_2(t) \to 0$ as $t \to \infty$. Continuing this process we deduce that $y_n(t) > 0$, $y_{n-1}(t) < 0, \ldots, y_2(t) > 0$, $y_1(t) > 0$ for $t \geqslant t_0$ and $y_k(t) \to 0$ as $t \to \infty$, $k = 3, 4, \ldots, n$. This proves the Lemma.

In order to characterize the behaviors of solutions, we may reformulate the Lemma as follows:

THEOREM 1. *If x is a nontrivial solution of* (1) (*or* (2)) *such that* $\lim_{t\to\infty} (x(t)/\alpha_2(t)) = 0$, *then either*

(a) *x is an oscillatory solution of* (1) (*or* (2)), *or else*

(b) *$x \geqslant 0$ ($\leqslant 0$) on $[t_1, \infty)$ for some $t_1 \geqslant t_0$ and x ($-x$) satisfies the conclusion of the Lemma. In particular x ($-x$) increases (decreases) monotonically on $[t_1, \infty)$.*

COROLLARY. *If x is a nontrivial solution of* (1) (*or* (2)) *such that $x(t) \to 0$ as $t \to \infty$, then x is oscillatory.*

Remarks. (1) If $h(t) = 0$, $a_i(t) = 1$, $i = 1, \ldots, n-1$, $f(t, x, \ldots) = q(t)\, x$, and the inequality in (1) (or (2)) bee replaced by equality, then [8, Theorem 1] and our Lemma are the same.

(2) If $h(t) = 0$ and $f(t, x, ...) = q(t) f(x[g(t)])$, then [1, Theorem 1] is included in our Lemma.

Let $1 \leqslant k \leqslant n-1$ and $t \in [t_0, \infty)$. We define

$$w_1(t) = \int_{t_0}^{t} \frac{1}{a_1(u)}\, du \qquad \text{and} \qquad w_k(t) = \int_{t_0}^{t} \frac{1}{a_k(u)}\, w_{k-1}(u)\, du.$$

THEOREM 2. *Let conditions* (3), (4), *and* (6)–(8) *hold, and let*

$$\int_{t_0}^{\infty} w_{n-1}(s) f(s, c, ..., c)\, ds = \infty, \tag{11}$$

where c is a positive constant; then every bounded solution of (1) (*or* (2)) *is oscillatory.*

Proof. We only consider (1). Let $x(t)$ be a nonoscillatory bounded solution of (1). Without loss of generality, we may assume that $x(t)$ and $x[g_i(t)]$ are positive for $t \geqslant t_0$ and $i = 1, 2, ..., m$. Hence

$$L_n x(t) + f(t, x(t), x[g_1(t)], ..., x[g_m(t)]) - h(t) \leqslant 0.$$

Let $y(t) = x(t) - \eta(t)$. Then

$$\begin{aligned} L_n y(t) + f(t, y(t) + \eta(t), y[g_1(t)] \\ + \eta[g_1(t)], ..., y[g_m(t)] + \eta[g_m(t)]) \leqslant 0, \end{aligned}$$

which implies $L_n y(t) < 0$. By our Lemma, we have

$$\sum_{j=1}^{n-1} (-)^{j-1} w_j(t)\, L_j y(t) \geqslant 0, \qquad t \geqslant t_1 \geqslant t_0.$$

It is easily verified that

$$\begin{aligned} y(t) \geqslant y(t_1) + \sum_{j=1}^{n-1} (-)^{j-1} w_j(t)\, L_j y(t) - \sum_{j=1}^{n-1} (-)^{j-1} w_j(t_1)\, L_j y(t_1) \\ + \int_{t_1}^{t} w_{n-1}(s) f(s, y(s) + \eta(s), ..., y[g_m(s)] + \eta[g_m(s)])\, ds. \end{aligned}$$

Thus for $t \geqslant t_1$, we obtain

$$y(t) \geqslant y(t_1) - \sum_{j=1}^{n-1} (-)^{j-1} w_j(t_1) L_j y(t_1) + \int_{t_1}^{t} w_{n-1}(s) f(s, y(s) + \eta(s), \ldots, y[g_m(s)] + \eta[g_m(s)]) \, ds. \tag{12}$$

Let $g_*(t) = \min\{g_1(t), g_2(t), \ldots, g_m(t)\}$. Since $x'(t) > 0$ and $y'(t) > 0$ for $t \geqslant t_1$, we have

$$\begin{aligned} x[g_i(t)] &\geqslant x[g_*(t)] = y[g_*(t)] + \eta[g_*(t)] \\ &\geqslant x[g_*(t_1)] = y[g_*(t_1)] + \eta[g_*(t_1)] \\ &= c > 0 \qquad \text{for} \quad t \geqslant t_1', \end{aligned}$$

and $i = 1, 2, \ldots, m$. Therefore (12) becomes

$$y(t) \geqslant y(t_1) - \sum_{j=1}^{n-1} (-)^{j-1} w_j(t_1) L_j y(t_1) + \int_{t_1}^{t} w_{n-1}(s) f(s, c, \ldots, c) \, ds.$$

Thus by (11), $\lim_{t \to \infty} y(t) = \infty$, a contradiction. This contradiction establishes our theorem.

THEOREM 3. *Let conditions* (3)–(8) *hold, and let*

$$\limsup_{t \to \infty} \frac{1}{\alpha_2(t)} \int^{t} w_{n-1}(s) f(s, c, \ldots, c) \, ds > 0, \tag{13}$$

where c is a positive constant. Then every nontrivial solution $x(t)$ of (1) (*or* (2)) *such that $x(t)/\alpha_2(t) \to 0$ as $t \to \infty$ is oscillatory.*

Proof. The proof of Theorem 3 is similar to that of Theorem 2 and is omitted.

Remarks. (1) The above results hold equally well for the case when $h(t) = 0$.

(2) If $h(t) = 0$, then [9, Theorem 1] and [1, Theorem 4] are included, respectively, in our Theorems 2 and 3.

EXAMPLE. The equation

$$\left(\frac{1}{t}\left(\frac{1}{t}\left(\frac{1}{t}x'\right)'\right)'\right)' + \frac{231}{16t^7} x^{\alpha}(t^{1/\alpha}) = 0, \qquad t > 0, \tag{14}$$

where α is the quotient of two odd positive integers, has the nonoscillatory unbounded solution $x(t) = \sqrt{t}$. All the conditions of Theorem 2 are satisfied, thus all bounded solutions of (14) are oscillatory. Only condition (13) of Theorem 3 is violated. We may add that the solutions of the equation

$$\left(\frac{1}{t}\left(\frac{1}{t}\left(\frac{1}{t}x^{\cdot}\right)^{\cdot}\right)^{\cdot}\right)^{\cdot}+\frac{231}{16t^7}x^{\alpha}(t^{1/\alpha})=e^{-t}\sin t, \qquad t>0, \tag{15}$$

α as above, have the same oscillation property as those of (14) for t sufficiently large because h satisfies condition (8).

For convenience of notation for any $t_0 \geqslant 0$ and $t \geqslant t_0$ we let

$$w(t)=\max_{1\leqslant i\leqslant m} a_i(t), \quad \gamma(t)=\int_{t_0}^{t}\frac{1}{a_1(s_1)}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{n-1}} h(s)\,ds\,ds_{n-1}\cdots ds_1.$$

THEOREM 4. *Let conditions* (3) *and* (6) *hold, and let*

$$\limsup_{t\to\infty}\,[\gamma(t)+kw(t)]=+\infty,$$

$$\liminf_{t\to\infty}\,[\gamma(t)+kw(t)]=-\infty.$$

for $t\in[0,\infty)$ *and every* $k\in(0,\infty)$. *Then every solution of* (1) (*or* (2)) *is oscillatory.*

Proof. The proof is similar to that of [7, Theorem 2.1] and is omitted.

THEOREM 5. *Let conditions* (3), (4), *and* (6) *hold, and*

$$\liminf_{t\to\infty}\int_0^t w_{n-1}(s)[-f(s,k,k,\ldots,k)+h(s)]\,ds=-\infty, \tag{16}$$

$$\limsup_{t\to\infty}\int_0^t w_{n-1}(s)[-f(s,-k,\ldots,-k)+h(s)]\,ds=\infty, \tag{17}$$

for every $k>0$. *Moreover, let*

$$\int h(t)\,dt=\phi(t)+C, \qquad \int\frac{\phi(t)}{a_{n-1}(t)}\,dt=\phi_1(t)+C, \tag{18}$$

where C *is any arbitrary constant,* ϕ *and* ϕ_1 *are two bounded functions with* ϕ *oscillatory.*

Then any bounded solution of (1) (*or* (2)) *is oscillatory.*

Proof. The proof is an adaptation of an argument developed by Kartsatos and Manougian [7, Theorem 2.4]. Again we only consider (1). Let

$x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we assume that $x(t)$, $x[g_i(t)]$, $i = 1, 2, \ldots, m$ are positive, $t \geqslant t_1 \geqslant t_0$. We also assume that there exists $L > 0$ such that $0 < x(t) \leqslant L$ and $0 < x[g_i(t)] \leqslant L$ for $t \geqslant t_1$, $i = 1, 2, \ldots, m$. We may take t_2 such that $\phi(t_2) = 0$. Now consider the transformation

$$M(t) = L_{n-1}x(t) - \phi(t), \qquad t \geqslant t_2.$$

Then we have

$$M^{\cdot}(t) = -f(t, x(t), \ldots, x[g_m(t)]) \leqslant 0.$$

Thus $M(t)$ is decreasing in $[t_2, \infty)$. Assume that $M(t_3) = -\lambda < 0$ for some $t_3 \geqslant t_2$. Then $M(t) \leqslant -\lambda$ in $[t_3, \infty)$. Thus

$$\frac{-\lambda}{a_{n-1}(t)} \geqslant \frac{M(t)}{a_{n-1}(t)} = (L_{n-2}x(t))^{\cdot} - \frac{\phi(t)}{a_{n-1}(t)}.$$

Thus

$$-\lambda \int_{t_3}^{t} \frac{1}{a_{n-1}(s)}\, ds \geqslant L_{n-2}x(t) - L_{n-2}x(t_3) - \phi_1(t) + \phi_1(t_3).$$

Since $\phi_1(t)$ is bounded, $\lim_{t \to \infty} L_{n-2}x(t) = -\infty$, a contradiction to the positivity of $x(t)$. Thus $M(t) \geqslant 0$ for $t \in [t_2, \infty)$. Now, by integration of the function

$$-(L_n x(t) - \phi^{\cdot}(t)) = f(t, x(t), x[g_1(t)], \ldots, x[g_m(t)])$$

we obtain

$$-M(t) + L_{n-1}x(t_2) = \int_{t_2}^{t} f(s, x(s), \ldots, x[g_m(s)])\, ds,$$

from which

$$L_{n-1}x(t_2) \geqslant \int_{t_2}^{t} f(s, x(s), \ldots, x[g_m(s)])\, ds.$$

Since t_2 is arbitrary, we have $L_{n-1}x(t) \geqslant 0$ for any $t \geqslant t_2$. It follows from our Lemma that

$$(-)^k L_k x(t) \leqslant 0 \quad \text{for} \quad t \geqslant \bar{t} \geqslant t_2, \quad k = 1, \ldots, n-1. \tag{19}$$

This implies that $x^{\cdot}(t) \geqslant 0$ for $t \geqslant \bar{t}$. Thus there exists $\bar{t}_1 \geqslant \bar{t}$ such that $x[g_i(t)] \geqslant K > 0$, $x(t) \geqslant K > 0$ for every $t \geqslant \bar{t}_1$, where K is a constant. Now differentiating the function $T(t) = w_{n-1}(t)\, L_{n-1}x(t)$, $t \geq \bar{t}_1$, and integrating from $\bar{t}_1$ to $t \geq \bar{t}_1$, we obtain

$$T(t) = T(\bar{t}_1) + \int_{\bar{t}_1}^{t} w_{n-1}(s)[-f(s, x(s),..., x[g_m(s)]) + h(s)]\, ds$$

$$+ \int_{\bar{t}_1}^{t} \frac{w_{n-2}(s)}{a_{n-1}(s)} L_{n-1}x(s)\, ds$$

$$\leq T(\bar{t}_1) + \int_{\bar{t}_1}^{t} w_{n-1}(s)[-f(s, K,..., K] + h(s)]\, ds$$

$$+ \int_{\bar{t}_1}^{t} w_{n-2}(s)(L_{n-2}x(s))^{\cdot}\, ds.$$

Now, applying (16), there exists a sequence $\{t_l\}$, $l = 1, 2,...$ such that $\lim_{l\to\infty} t_l = \infty$ and

$$\lim_{l\to\infty} \int_{t_1}^{t_l} w_{n-1}(s)[-f(s, K, K,..., K) + h(s)]\, ds = \infty.$$

This implies that

$$\int_{\bar{t}_1}^{\infty} w_{n-2}(s)(L_{n-2}x(s))^{\cdot}\, ds = \infty \tag{20}$$

because $L_{n-1}x(t) \geq 0$, $t \in [\bar{t}_1, \infty)$. Successive integration by parts of (20) and by the use of (19) we obtain

$$\begin{aligned}\int_{\bar{t}_1}^{\infty} w_{n-i}(s)(L_{n-i}x(s))^{\cdot}\, ds &= +\infty, \qquad \text{if} \quad i = 2, 4,..., n,\\ &= -\infty, \qquad \text{if} \quad i = 3, 4,..., n-1.\end{aligned}$$

Thus if $i = n$, $\lim_{t\to\infty} x(t) = \infty$, a contradiction to the fact that x is a bounded function.

THEOREM 6. *Let conditions* (3)–(6) *and* (18) *hold, and*

$$\liminf_{t\to+\infty} \frac{1}{\alpha_2(t)} \int_0^t w_{n-1}(s)[-f(s, k,..., k) + h(s)]\, ds < 0, \tag{21}$$

$$\limsup_{t\to+\infty} \frac{1}{\alpha_2(t)} \int_0^t w_{n-1}(s)[-f(s, -k,..., -k) + H(s)]\, ds > 0, \tag{22}$$

for every $k > 0$. Then every nontrivial solution $x(t)$ of (1) (*or* (2)) *such that $x(t)/\alpha_2(t) \to 0$ as $t \to \infty$ is oscillatory.*

Proof. The proof is similar to the proof of Theorem 5 and is omitted.

THEOREM 7. *Let conditions* (3), (4), (6), *and* (18) *hold, and*

$$\liminf_{t\to\infty}\int_0^t [-f(s, k,..., k) + h(s)]\, ds = -\infty,$$

$$\limsup_{t\to\infty}\int_0^t [-f(s, -k,..., -k) + h(s)]\, ds = +\infty.$$

Then every solution of (1) (*or* (2)) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality we may assume that $x(t)$, $x[g_i(t)]$, $i = 1,..., m$ are positive for $t \geqslant t_1 \geqslant t_0$. Thus it follows from the proof of Theorem 5 that $L_{n-1}x(t) \geqslant 0$ for every $t \geqslant t_2$, for some $t_2 \geqslant t_1$. This implies that there exists a constant $K > 0$ such that $x(t) \geqslant K$, $x[g_i(t)] \geqslant K$ for every $t \geqslant t_2$. This follows from the fact that $x(t)$ is positive, n is even, and $L_{n-1}x(t) \geqslant 0$, $x'(t) \geqslant 0$ for all large t. Thus integrating (1) once we get

$$\begin{aligned} L_{n-1}x(t) &= L_{n-1}x(t_2) + \int_{t_2}^t [-f(s, x(s),..., x[g_m(s)]) + h(s)]\, ds \\ &\leqslant L_{n-1}x(t_2) + \int_{t_2}^t [-f(s, K,..., K) + h(s)]\, ds, \end{aligned}$$

which implies that $\liminf_{t\to\infty} x(t) = -\infty$, a contradiction. A similar argument holds for $x(t)$ eventually negative and this completes the proof.

Remarks. (1) If $a_i(t) = 1$, $i = 1, 2,..., n-1$, then [7, Theorems 2.4 and 2.6] are included, respectively, in our Theorems 6 and 7.

(2) We believe that our Theorems 3 and 6 are new oscillation criteria for (1) (or (2)).

REFERENCES

1. S. R. GRACE AND B. S. LALLI, On oscillation of solutions of nth order delay differential equation, *J. Math. Anal. Appl.* **91** (1983), 328–339.
2. W. E. MAHFOUD, Characterization of oscillation of solutions of the delay equation $x^{(n)} + a(t)f(x[q(t)]) = 0$, *J. Differential Equations* **28** (1978), 437–451.
3. W. E. MAHFOUD, Oscillation and asymptotic behavior of solutions of nth order nonlinear delay differential equation, *J. Differential Equations* **24** (1977), 617–627.
4. A. G. KARTSATOS,On the maintenance of oscillations under the effect of a small forcing term, *J. Differential Equations* **10** (1970), 355–363.
5. A. G. KARTSATOS, Maintenance of oscillations under the effect of a periodic forcing term. *Proc. Amer. Math. Soc.* **33** (1972), 377–383.

6. A. G. KARTSATOS, Recent results on oscillation of solutions of forced and perturbed differential equations of even order *in* "Proceedings, NSF-CBNS Reg. Conference, Stability of Dynamical Systems, Mississippi State University, 1977" (J. R. Graef, Ed.), Dekker, New York, 1977.
7. A. G. KARTSATOS AND M. N. MANOUGIAN, Further results on oscillation of functional differential equation, *J. Math. Anal. Appl.* **53** (1976), 28–37.
8. W. J. KIM, Monotone and oscillatory solutions of $y(n) + py = 0$, *Proc. Amer. Math. Soc.* **62** (1977), 77–82.
9. D. L. LOVELADY, On oscillatory behavior of bounded solutions of higher order differential equations, *J. Differential Equations* **19** (1975), 167–175.